TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 166, April 1972

SLICING THEOREMS FOR n-SPHERES IN EUCLIDEAN $(n+1)$-SPACE

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Abstract. This paper describes conditions on the intersection of an n-sphere Σ in Euclidean $(n+1)$-space E^{n+1} with the horizontal hyperplanes of E^{n+1} sufficient to determine that the sphere be nicely embedded. The results generally are pointed towards showing that the complement of Σ is 1-ULC (uniformly locally 1-connected) rather than towards establishing the stronger property that Σ is locally flat. For instance, the main theorem indicates that $E^{n+1}-\Sigma$ is 1-ULC provided each nondegenerate intersection of Σ and a horizontal hyperplane be an $(n-1)$-sphere bicollared both in that hyperplane and in Σ itself $(n\neq 4)$.

1. **Introduction.** Much of the literature that treats the problem of determining properties of the embedding of an object Σ in E^n from information about the intersection of Σ with the horizontal hyperplanes of E^n focuses on the case $n=3$. Such a problem first arose in this dimension when J. W. Alexander [1] suggested that a 2-sphere in E^3 might be embedded just as a round sphere if each of its intersections with the horizontal planes were either a point or a simple closed curve. Recently Eaton [11] and Hosay [12] showed this to be true. After generalizations by Loveland [15] and Jensen [13], Cannon proved that the same property is held by any 2-sphere Σ in E^3 such that no intersection of Σ with a horizontal plane has a degenerate component [7].

For higher dimensions Bryant has proved that a k-dimensional compact set X in E^n, where $n-k\geqq 3$, has a 1-ULC complement if the complement of X with respect to each member of some dense subset of the horizontal hyperplanes of E^n is 1-ULC [5].

The main results of this paper are found in §5, where it is shown that a closed n-manifold Σ topologically embedded in E^{n+1} $(n\neq 4)$ is nice (meaning, $E^{n+1}-\Sigma$ is 1-ULC) if each nondegenerate intersection Σ_t of Σ and a horizontal, n-dimensional hyperplane in E^{n+1} is a PL $(n-1)$-manifold bicollared in that hyperplane and nice in Σ $(\Sigma-\Sigma_t$ is 1-ULC). Most of the techniques required are consigned to §4. In §6 other generalizations to the results mentioned in the first paragraph are given, the methods for which are taken from [7] and [12]. In addition, we describe

Received by the editors March 15, 1971.

AMS 1970 *subject classifications*. Primary 57A45, 57A50; Secondary 57A15.

Key words and phrases. Horizontal hyperplane of Euclidean space, bicollared submanifold, locally flat submanifold, locally simply connected, homeomorphic approximation, topological embedding.

(1) Research was supported in part by NSF Grant GP-19966.

in §3 some methods, similar to but weaker than those of the three-dimensional case (see [3]), for improving mappings of a disk into the closure of a complementary domain of an n-manifold in an $(n+1)$-manifold.

2. **Definitions and notation.** An *n-manifold* is a separable metric space which is locally homeomorphic to E^n; thus, the term manifold is reserved for manifolds without boundary. For simplicity we shall assume all manifolds to be connected, but they need not be compact or triangulated. A manifold that is compact (and without boundary) is said to be *closed.*

A subset S of a metric space is called an *ε-subset* if and only if the diameter of S, written diam S, is less than ε.

Suppose f and g are maps of a space X into a space Y that has a metric ρ. The symbol $\rho(f, g)<\varepsilon$ means that $\rho(f(x), g(x))<\varepsilon$ for each x in X. The maps f and g are said to be *ε-homotopic* (*ε-isotopic*) if and only if there exists a homotopy (isotopy) h_t sending X into Y such that $h_0=f$, $h_1=g$ and $\rho(h_s, h_t)<\varepsilon$ for all s, t in $[0, 1]$.

A map f of the metric space Y into a subset A is an *ε-map* if and only if $\rho(y, f(y))<\varepsilon$ for each $y \in Y$.

The symbol Δ^2 denotes a 2-simplex fixed throughout this paper and $\partial\Delta^2$ denotes its boundary. Given a triangulation R of Δ^2, we use $R^{(i)}$ to denote the i-skeleton of R $(i=0, 1)$.

For any point p in a metric space S and any positive number δ, $N_\delta(p)$ denotes the set of points in S whose distance from p is less than δ.

Let A denote a subset of a metric space X and p a limit point of A. We say that A is *locally simply connected at p*, written 1-LC at p, if and only if for each $\varepsilon>0$ there is a $\delta>0$ such that each map of $\partial\Delta^2$ into $A \cap N_\delta(p)$ can be extended to a map of Δ^2 into $A \cap N_\varepsilon(p)$. Furthermore, we say that A is *uniformly locally simply connected*, written 1-ULC, if for each $\varepsilon>0$ there is a $\delta>0$ such that each map of $\partial\Delta^2$ into a δ-subset of A can be extended to a map of Δ^2 into an ε-subset of A.

In the same context we say that A is *locally arcwise connected at p*, or 0-LC at p, if and only if for each $\varepsilon>0$ there exists a $\delta>0$ such that any map of ∂I (where $I=[0, 1]$) into $A \cap N_\delta(p)$ extends to a map of I into $A \cap N_\varepsilon(p)$. We define analogously the uniform condition 0-ULC.

For a subset U of a space S we use the symbol Cl U to denote the closure of U and Bd U to denote the (topological) boundary of U in S.

Let Σ be an n-manifold embedded in the interior of an $(n+1)$-manifold M as a closed subset, and let U be an open subset of $M-\Sigma$. Then Σ is *collared from U* if and only if there exists a homeomorphism g of $\Sigma \times I$ into Cl U such that $g(s, 0)=s$ for each s in Σ. Similarly, Σ is *bicollared in M* if and only if there exists a homeomorphism h of $\Sigma \times [-1, 1]$ into M such that $h(s, 0)=s$ for each s in Σ. In addition, Σ is *locally flat in M* if and only if each point s of Σ has a neighborhood N (relative to M) such that $N \cap \Sigma$ is bicollared in N.

Let n be a positive integer. For each real number t define E_t, the hyperplane of E^{n+1} at the t-level, as $\{(x_1, \ldots, x_{n+1}) \in E^{n+1} \mid x_{n+1}=t\}$. For any subset Σ of E^{n+1} we define Σ_t as $\Sigma \cap E_t$, and for any set C of real numbers we define $\Sigma(C)$ as $\bigcup \{\Sigma_t \mid t \in C\}$; however, for an interval (a, b) we simplify $\Sigma((a, b))$ to $\Sigma(a, b)$.

3. Altering maps of a disk.

LEMMA 3.1. *Suppose Σ is an n-manifold embedded in the interior of an $(n+1)$-manifold M as a closed separating subset, U a component of $M-\Sigma$, X a closed subset of Σ such that* Cl $U-X$ *is* 1-*LC at each point of X, R a triangulation of Δ^2, T a subdivision of R, and F a map of Δ^2 into* Cl U *such that $F(|R^{(1)}|) \subset U$.*

Then for each $\varepsilon>0$ there exists a map G of Δ^2 into Cl U *such that*

(a) $G \mid |R^{(1)}| = F| \; |R^{(1)}|$,

(b) $\rho(F, G) < \varepsilon$,

(c) $G(|T^{(1)}|) \subset U$,

(d) $G(\Delta^2) \cap X = \varnothing$.

Proof. Note that, since $F(\Delta^2)$ is compact, there exist a neighborhood V of $F(\Delta^2)$ (relative to Cl U) and a positive number δ such that any map of $\partial\Delta^2$ into a δ-subset of $V-X$ can be extended so as to send Δ^2 into an $(\varepsilon/3)$-subset of Cl $U-X$.

Now we simply modify F, beginning with the 0-skeleton of T and working up. In case $v \in T^{(0)}$ and $F(v) \in \Sigma$, define $G(v)$ as a point of $U \cap V$ very close to $F(v)$; when $F(v) \notin \Sigma$, define $G(v)=F(v)$. Since U is 0-LC at each point of Σ [17, Theorem II.5.35], then, for each 1-simplex σ of T, G can be extended along σ in such a way that $G(\sigma) \subset U \cap V$ and $\rho(G|\sigma, F|\sigma) < \varepsilon/3$; in case $\sigma \subset |R^{(1)}|$, define $G|\sigma = F|\sigma$. Because first we could have subdivided T (if necessary), we can assume that, for each 2-simplex τ of T, Diam $F(\tau) < \varepsilon/3$ and Diam $G(\partial\tau) < \delta$. According to the previous paragraph, G can be extended over τ into an $\varepsilon/3$-subset of Cl $U-X$. It follows easily that $\rho(F, G) < \varepsilon$.

THEOREM 3.2. *Suppose Σ is an n-manifold embedded in the interior of an $(n+1)$-manifold M as a closed separating subset, U a component of $M-\Sigma$, and f a map of Δ^2 into* Cl U *with $f(\partial\Delta^2) \subset U$. Suppose $\{X^i\}$ is a countable collection of closed subsets of Σ such that* Cl $U-X^i$ *is* 1-*LC at each point of X^i $(i=1, 2, \ldots)$. Then for each $\varepsilon>0$ there exists a map g of Δ^2 into* Cl U *such that*

(1) $g|\partial\Delta^2 = f|\partial\Delta^2$,

(2) $\rho(f, g) < \varepsilon$,

(3) $g^{-1}(\Sigma \cap g(\Delta^2))$ *is* 0-*dimensional,*

(4) $g(\Delta^2) \cap X^i = \varnothing$ *for* $i=1, 2, \ldots$.

The proof follows from routine applications of Lemma 3.1.

This result yields the following corollary, which can be regarded as a very weak version of [3, Theorem 4.2].

COROLLARY 3.3. *Suppose* Σ *is an n-manifold embedded in the interior of an* $(n+1)$*-manifold M as a closed separating subset, U a component of* $M-\Sigma$, *and f a map of* Δ^2 *into* Cl U *such that* $f(\partial\Delta^2)\subset U$. *Then for each* $\varepsilon>0$ *there exists a map g of* Δ^2 *into* Cl U *such that* (i) $g|\partial\Delta^2=f|\partial\Delta^2$, (ii) $\rho(f,g)<\varepsilon$, *and* (iii) $g^{-1}(\Sigma\cap g(\Delta^2))$ *is* 0*-dimensional.*

4. **Embedding Cartesian products in** E^{n+1}. Let n denote a fixed positive integer. Obviously there exists a countable collection $\mathscr{M}$ of closed, PL $(n-1)$-manifolds such that any closed, PL $(n-1)$-manifold is homeomorphic to some member of $\mathscr{M}$. (In fact, this holds even without the PL hypothesis [8].)

In this section Σ will denote an n-manifold embedded in E^{n+1} as a closed subset, U a component of $E^{n+1}-\Sigma$, and (a, b) an interval of real numbers such that, for each $t\in(a, b)$, Σ_t is homeomorphic to some member of $\mathscr{M}$ and is collared from U_t. Let $(a, b)_M$ be the set of all t in (a, b) such that Σ_t is homeomorphic to M, where $M\in\mathscr{M}$. For each such M and each $t\in(a, b)_M$ define an embedding of $M\times[-1, 1]$ into Cl U_t such that $\lambda_t(M\times\{-1\})=\Sigma_t$. Topologize the set $\{\lambda_t \mid t\in (a, b)_M\}$ by the sup-norm metric in E^{n+1}, producing a separable metric space.

As suggested by Bryant [4] (and in the unpublished work of Bing [2]), one can easily establish the following lemma.

LEMMA 4.1. *There exists a countable subset* $\mathscr{D}$ *of* (a, b) *such that to each t in* $(a, b)-\mathscr{D}$ *there correspond two sequences* $\{s(i)\}$ *and* $\{u(i)\}$ *of real numbers, with* $a<s(i)<t<u(i)<b$, *such that each of the associated sequences* $\{\lambda_{s(i)}\}$ *and* $\{\lambda_{u(i)}\}$ *converges* (*homeomorphically*) *to* λ_t.

Throughout the rest of §§4 and 5, $\mathscr{D}$ will denote the subset of (a, b) described in Lemma 4.1, and n will denote a fixed integer other than 4.

The following result can be established by adding simple epsilonics to the proof of Borsuk's theorem (see [10, Theorem 10.2]).

LEMMA 4.2. *Let A be an ANR embedded as a closed subset of a metric space X,* $\varepsilon>0$, *and* $f\colon X\to A$ *an* ε*-map such that* $f|A$ *is* ε*-homotopic* (*in A*) *to the identity map. Then there exists a* 2ε*-retraction of X onto A.*

LEMMA 4.3. A. *If* $t\in(a, b)_M-\mathscr{D}$, *then there exists a homeomorphism h of* $M\times[-1, 1]$ *onto a subset* A_t *of* Cl U *such that*

$$\Sigma\cap A_t = h(M\times\{-1\})\cup h(M\times\{1\}) = \Sigma_s\cup\Sigma_u,$$

where $a<s<t<u<b$ *and* $A_t\cap E_z=\varnothing$ *for z not in* $[s, u]$.

B. *For any such* A_t, *let* X_t *denote the closure of the component of* Cl $U-A_t$ *containing* Σ_t. *Then, for each* $\varepsilon>0$, A_t *can be obtained so that there exists an* ε*-retraction of* X_t *onto* A_t.

Proof. Fix a point t of $(a, b)_M-\mathscr{D}$. By restricting $[-1, 1]$ to a subinterval containing -1, if necessary, we may assume that diam $\lambda_t(\{p\}\times[-1, 1])$ is less than

$\varepsilon/18$ for each p in M. Since $\lambda_t(M\times\{0\})$ is an ANR, the obvious retraction of $\lambda_t(M\times[-1,1])$ onto $\lambda_t(M\times\{0\})$ can be extended over a neighborhood N of $\lambda_t(M\times[-1,1])$ to an $\varepsilon/18$-retraction R of N onto $\lambda_t(M\times\{0\})$. Furthermore, N can be chosen as a product $N'\times(s',u')\subset E^n\times E^1$ where N' is a bounded open subset of E^n and $N\cap\Sigma=\Sigma(s',u')$.

It is sufficient to describe the homeomorphism h of $M\times[-1,1]$ onto some (as yet undefined) A_t subject to the following conditions:

(1) $X_t\subset N$,

(2) diam $h(\{p\}\times[-1,1])<\varepsilon/6$ for each p in M,

(3) $h(p,0)=\lambda_t(p,0)$ for each p in M.

Let g denote the map of A_t to $h(M\times\{0\})$ sending each $h(\{p\}\times[-1,1])$ to $h(p,0)$. The product structure on A_t will provide the natural guide for defining an $\varepsilon/6$-homotopy $G_s\colon A_t\to A_t$ between g and the identity map. Note that condition (2) above and the definition of R imply that

$$\operatorname{diam} Rh(\{p\}\times[-1,1]) < \varepsilon/3$$

for each p in M. Thus, RG_s will be an $\varepsilon/3$-homotopy between $Rg=g$ and R. This means that $R|A_t$ will be $\varepsilon/2$-homotopic in A_t to the identity map, and part B of this lemma will be a consequence of Lemma 4.2.

Let δ_t denote the distance from $\lambda_t(M\times\{0\})$ to $\Sigma\cap(E^{n+1}-N)$. It follows from [18] (see also [16, Lemma 5]) in case $n>4$ and from [14, Lemma 4] or [9, Theorem 8.2] in case $n=3$ that there exists a $d_t>0$ such that any locally flat n-manifold in E_t homeomorphically within d_t of Σ_t is δ_t-isotopic to Σ_t in E_t. According to Lemma 4.1 there exist real numbers s and u, with $s'<s<t<u<u'$, such that λ_s and λ_u are homeomorphically within d_t of λ_t. If p_t denotes the map projecting $E^n\times E^1$ onto $E^n\times\{t\}$, then $p_t\lambda_s(M\times\{0\})$ and $p_t\lambda_u(M\times\{0\})$ are each δ_t-isotopic to $\lambda_t(M\times\{0\})$ in E_t. By lifting this isotopy through the levels E_r $(s\leqq r\leqq u)$, we construct an embedding h of $M\times[-\frac{1}{2},\frac{1}{2}]$ into $U\cap N$ such that condition (3), as well as the following conditions, holds:

(4) $h(M\times\{-\frac{1}{2}\})=\lambda_s(M\times\{0\})$,

(5) $h(M\times\{\frac{1}{2}\})=\lambda_u(M\times\{0\})$,

(6) for each $w\in[-\frac{1}{2},\frac{1}{2}]$, there exists a distinct $z\in[s,u]$ such that $h(M\times\{w\})\subset E_z$,

(7) diam $h(\{p\}\times[-\frac{1}{2},\frac{1}{2}])<\varepsilon/18$ for each p in M.

Since both λ_s and λ_u are homeomorphically close to λ_t, we may assume s and u were chosen so that

(8) diam $\lambda_s(\{p\}\times[-1,0])<\varepsilon/18$ for each p in M,

(9) diam $\lambda_u(\{p\}\times[-1,0])<\varepsilon/18$ for each p in M.

Now $h|M\times[-\frac{1}{2},\frac{1}{2}]$ can be extended to a homeomorphism h of $M\times[-1,1]$ onto

$$A_t = \lambda_s(M\times[-1,0])\cup h(M\times[-\tfrac{1}{2},\tfrac{1}{2}])\cup\lambda_u(M\times[-1,0]).$$

To verify that $X_t \subset N$, observe that N has connected boundary, as does X_t (where Bd X_t is taken in E^{n+1}, not in Cl U). Since Bd $X_t \subset N$ by construction and since both X_t and N are bounded, X_t must be a subset of N. The only unverified requirement on the construction, condition (2), follows easily from (7)–(9).

ADDENDUM. Suppose $\varepsilon > 0$, X_t and A_t satisfy the conclusions of Lemma 4.3B, and Z is a compact subset of X_t. Then there exists an ε-map of Z into A_t such that $f(Z) \cap \text{Bd } A_t \subset Z$ and $f|Z \cap A_t = \text{identity}$.

Proof. Follow the ε-retraction of X_t onto A_t by a small homeomorphism g of A_t into $(Z \cap \text{Bd A}_t) \cup \text{Int } A_t$ such that $g|Z \cap A_t = \text{identity}$.

5. Submanifolds of E^{n+1} whose levels are twice bicollared.

The basic result in this section is the following application of Lemma 4.3.

THEOREM 5.1. *Let Σ denote an n-manifold embedded in E^{n+1} $(n \neq 4)$ as a closed subset, U a component of $E^{n+1} - \Sigma$, and (a, b) an interval such that for each $t \in (a, b)$* (i) *Σ_t is a closed, PL $(n-1)$-manifold that is collared from U_t and* (ii) Cl *$U - \Sigma_t$ is 1-LC at each point of Σ_t. Then U is 1-LC at each point of $\Sigma(a, b)$.*

Proof. Suppose $f: \Delta^2 \to \text{Cl } U$ is a map such that $f(\partial\Delta^2) \subset U$ and $f(\Delta^2) \cap \Sigma \subset \Sigma(a, b)$. Let ε be a positive number less than both $\rho(f(\Delta^2), \Sigma_a \cup \Sigma_b)$ and $\rho(f(\partial\Delta^2), \Sigma)$, and let $\mathscr{D}$ denote the countable subset of (a, b) described in Lemma 4.1. Then by Theorem 3.2 there exists a map $f_0: \Delta^2 \to \text{Cl } U$ such that

(1) $f_0|\partial\Delta^2 = f|\partial\Delta^2$,

(2) $\rho(f, f_0) < \varepsilon/3$,

(3) $f_0(\Delta^2) \cap \Sigma_d = \varnothing$ for each d in D.

For each $t \in (a, b) - \mathscr{D}$, application of Lemma 4.3 yields an $\varepsilon/3$-retraction r_t of the X_t onto the A_t associated (by Lemma 4.3B) with t and $\varepsilon/3$, where $\Sigma \cap A_t = \Sigma_{s(t)} \cup \Sigma_{u(t)}$. Let π denote the projection of $E^n \times E^1$ onto the second factor. Then $\pi(\Sigma \cap f_0(\Delta^2))$, a subset of $(a, b) - \mathscr{D}$, is covered by the collection $\mathscr{C}$ of open intervals $\mathscr{C} = \{(s(t), u(t)) \mid t \in \pi(\Sigma \cap f_0(\Delta^2))\}$, from which we can extract a finite subcovering $\mathscr{F}$ of $\pi(\Sigma \cap f_0(\Delta^2))$. After eliminating unnecessary elements of $\mathscr{F}$, we can regard $\mathscr{F}$ as the union of two (finite) collections $\mathscr{F}_1$ and $\mathscr{F}_2$ such that for $i = 1, 2$ no two elements of $\mathscr{F}_i$ intersect. Let F_1 and F_2 denote the underlying point sets of $\mathscr{F}_1$ and $\mathscr{F}_2$ respectively.

In case $(s(t), u(t)) \in \mathscr{F}_1$, the addendum to Lemma 4.3 implies the existence of an $\varepsilon/3$ map R_t of $X_t \cap f_0(\Delta^2)$ into A_t such that

(4) $R_t(X_t \cap f_0(\Delta^2)) \cap \text{Bd } A_t \subset X_t \cap f_0(\Delta^2)$,

(5) $R_t|f_0(\Delta^2) \cap A_t = \text{identity}$.

Define a map $f_1: \Delta^2 \to \text{Cl } U$ by the rule

$$\begin{aligned} f_1(x) &= R_t f_0(x), \quad \text{if } f_0(x) \in X_t \text{ and } (s(t), u(t)) \in \mathscr{F}_1, \\ &= f_0(x), \quad \text{otherwise.} \end{aligned}$$

It follows from (5) and the fact that the X_t's considered are pairwise disjoint that f_1 is well defined and continuous. Note that

(6) $f_1|\partial\Delta^2=f_0|\partial\Delta^2$,

(7) $\rho(f_0, f_1)<\varepsilon/3$,

(8) $\pi(\Sigma \cap f_1(\Delta^2))\subset F_2$.

Similarly, in case $(s(t), u(t)) \in \mathscr{F}_2$, the addendum to Lemma 4.3 implies the existence of an $\varepsilon/3$-map R_t of $X_t \cap f_1(\Delta^2)$ into A_t such that

(9) $R_t(X_t \cap f_1(\Delta^2)) \cap \text{Bd } A_t=\varnothing$,

(10) $R_t|f_1(\Delta^2) \cap A_t=\text{identity}$.

Define $f_2: \Delta^2 \to \text{Cl } U$ by the rule

$$\begin{aligned} f_2(x) &= R_t f_1(x), \quad \text{if } f_1(x) \in X_t \text{ and } (s(t), u(t)) \in \mathscr{F}_2, \\ &= f_1(x), \qquad \text{otherwise.} \end{aligned}$$

As before, f_2 is a continuous function satisfying

(11) $f_2|\partial\Delta^2=f_1|\partial\Delta^2$,

(12) $\rho(f_1, f_2)<\varepsilon/3$,

(13) $f_2(\Delta^2) \cap \Sigma=\varnothing$.

Since $f_2|\partial\Delta^2=f|\partial\Delta^2$ and $\rho(f, f_2)<\varepsilon$, it follows immediately that U is 1-LC at each point of $\Sigma(a, b)$.

REMARKS. Although condition (ii) clearly is a necessary hypothesis for Theorem 5.1, one questions whether it might be superfluous. Even without this condition it follows from [4], by means of the trick emerging here in Lemma 4.1, that Cl $U-\Sigma_t$ is 1-LC at each point of Σ_t for all but at most countably many points t in (a, b), but this fact, obviously, is no help. By attacking the problem differently in the next section, we shall prove, under the hypotheses of Theorem 5.1 without condition (ii), that U is 1-LC at many points of Σ (see Corollary 6.2).

Without much extra effort one can prove the following slightly stronger version of Theorem 5.1. Statements of the other results in this section can be altered in a similar manner.

THEOREM 5.2. *Let Σ denote an n-manifold embedded in E^{n+1} $(n\neq 4)$ as a closed subset, U a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that* (i) *for each $t \in (a, b)$, Cl $U-\Sigma_t$ is 1-LC at each point of Σ_t and* (ii) *for all but countably many points t in (a, b), Σ_t is a closed, PL $(n-1)$-manifold that is collared from U_t. Then U is 1-LC at each point of $\Sigma(a, b)$.*

Proof. Simply incorporate those countably many t's of (a, b) that fail to satisfy condition (ii) into the set $\mathscr{D}$ and reapply the proof of Theorem 5.1.

THEOREM 5.3. *Let Σ denote an n-manifold embedded in E^{n+1} $(n\neq 4)$ as a closed subset, U a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that for each $t \in (a, b)$* (i) *Σ_t is a closed, PL $(n-1)$-manifold that is collared from U_t and* (ii) *$\Sigma-\Sigma_t$ is 1-LC at each point of Σ_t. Then U is 1-LC at each point of $\Sigma(a, b)$.*

Proof. Let p be a point of Σ_t and $\varepsilon>0$. There exists a $\delta>0$ such that any loop in $N_\delta(p) \cap (\Sigma-\Sigma_t)$ is contractible in $N_\varepsilon(p) \cap (\Sigma-\Sigma_t)$. Since U is locally 1-connected at p in the homology sense [17, Theorem II, 5.35], it follows from [6, Proposition 3.3] that there exists an $\alpha>0$ such that any loop in $N_\alpha(p) \cap U$ is contractible in $N_\delta(p) \cap (E^{n+1}-\Sigma_t)$. By cutting off such a contraction in $\Sigma-\Sigma_t$ one can show that each loop in $N_\alpha(p) \cap U$ is contractible in $N_\varepsilon(p) \cap (\mathrm{Cl}\ U-\Sigma_t)$. Using this property one easily can prove that $\mathrm{Cl}\ U-\Sigma_t$ is 1-LC at p. Hence, Theorem 5.1 gives the desired result.

THEOREM 5.4. *Let Σ denote a closed n-manifold in E^{n+1} ($n\neq 4$), U a component of $E^{n+1}-\Sigma$, and $[a, b]$ the interval such that $\Sigma=\Sigma([a, b])$. Suppose that for each t in (a, b), Σ_t is a closed PL $(n-1)$-manifold that is collared from U_t and that for each t in $[a, b]$, $\mathrm{Cl}\ U-\Sigma_t$ is 1-ULC. Then U is 1-ULC.*

Proof. Obviously U is 1-LC at points of $\Sigma(a, b)$. Furthermore, by hypothesis, any small loop in U near a point of Σ_a is contractible in a small subset of $\mathrm{Cl}\ U-\Sigma_a$. According to Theorem 3.2 such a contraction can be modified slightly so that the range of the resulting map is a small subset of U. Thus, U is 1-LC at points of Σ_a. The same argument applies to points of Σ_b. Consequently, U is 1-ULC.

Using Theorem 5.4 one can extend results of [11] and [12] to higher dimensions in the following ways.

COROLLARY 5.5. *Suppose Σ is a closed n-manifold in E^{n+1} ($n\neq 4$) such that* (i) $\Sigma=\Sigma([-1, 1])$, (ii) $\Sigma-(\Sigma_{-1} \cup \Sigma_1)$ *is 1-ULC, and* (iii) *for each $t \in (-1, 1)$, Σ_t is a closed, PL $(n-1)$-manifold bicollared in E_t and $\Sigma-\Sigma_t$ is 1-ULC. Then $E^{n+1}-\Sigma$ is 1-ULC.*

Proof. For either component U of $E^{n+1}-\Sigma$ and each t in $[-1, 1]$, the proof of Theorem 5.3 indicates that $\mathrm{Cl}\ U-\Sigma_t$ is 1-ULC. Although the hypotheses of Proposition 3.3 of [6] do not apply when $t=\pm 1$, the argument there can be used to establish the property employed in proving Theorem 5.3, namely, for each $\delta>0$ there exists an $\alpha>0$ such that each α-loop in U is contractible in a δ-subset of $E^{n+1}-\Sigma_t$ ($t\neq \pm 1$).

COROLLARY 5.6. *Suppose Σ is an n-sphere in E^{n+1} ($n\neq 4$) such that both Σ_1 and Σ_{-1} are points and, for each t in $(-1, 1)$, Σ_t is an $(n-1)$-sphere bicollared in E_t and $\Sigma-\Sigma_t$ is 1-ULC. Then $E^{n+1}-\Sigma$ is 1-ULC.*

From Theorem 9 of [16] we obtain the following flatness conditions.

THEOREM 5.7. *Suppose Σ is a closed PL n-manifold in E^{n+1} ($n\geqq 5$) satisfying the hypothesis of Corollary 5.5. Then Σ is locally flat if and only if Σ can be homeomorphically approximated by locally flat manifolds.*

THEOREM 5.8. *Let Σ denote the boundary of an $(n+1)$-cell B in E^{n+1} ($n\geqq 5$) and U the complement of B. Suppose* (i) $\Sigma=\Sigma([a, b])$, (ii) *for each $t \in (a, b)$, Σ_t is a PL $(n-1)$-manifold that is collared from U_t, and* (iii) *for each $t \in [a, b]$, $\mathrm{Cl}\ U-\Sigma_t$ is 1-ULC. Then Σ is locally flat.*

6. **Submanifolds of E^{n+1} whose levels satisfy 1-ULC conditions.** In this section we give sufficient conditions, in the spirit of Cannon's work in E^3 [7], for a complementary domain of a closed n-manifold in E^{n+1} to be 1-ULC. In place of the hypothesis typical of the results found in §5 that the Σ_t's be collared PL manifolds stands the weakened hypothesis that the U_t's be 1-ULC, together with strong restrictions on the embeddings of the Σ_t's in Σ. As one advantage of this approach, the case $n=4$ need not be excluded.

THEOREM 6.1. *Suppose that Σ is a closed n-manifold in E^{n+1}, Z a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that* (i) $\Sigma_t=\text{Bd } Z_t$ *and* (ii) Z_t *is both* 0-*ULC and* 1-*ULC for each t in (a, b). Then (a, b) contains a dense G_δ-subset G such that Z is* 1-*LC at each point of* $\Sigma(G)$.

Proof. Equivalently we shall show that the set F of levels t in (a, b) at which Σ_t contains a point q such that Z fails to be 1-LC at q is a 0-dimensional F_σ-set.

For each positive integer n let X_n denote the set of points x in Σ such that for no neighborhood V of x is every loop of $V \cap Z$ contractible in a $(1/n)$-subset of Z. Then X_n is a compact subset of Σ, and therefore $\pi(X_n)$ is a closed subset of E^1, where π denotes the map projecting $E^n \times E^1$ onto the second factor. Define F_n as $(a, b) \cap \pi(X_n)$. Obviously $F=\bigcup F_n$. Hence, we only need show F_n to be 0-dimensional.

Suppose to the contrary that some F_n contains a subinterval $[a', b']$ of (a, b). By the Baire Category Theorem $[a', b']$ then contains a subinterval $[c, d]$ such that corresponding to some dense subset Y of $[c, d]$ there exists a positive number δ with the property that each δ-loop in Z_t is null homotopic in a $(1/3n)$-subset of Z_t $(t \in Y)$. To reach the required contradiction we shall apply Hosay's argument [12] to prove that each point q of $\Sigma(c, d)$ has a neighborhood V such that every loop in $V \cap Z$ is null-homotopic in a $(1/n)$-subset of Z.

Given any such point q let U be a round open ball in E^{n+1} containing q of diameter less than $\min\{\delta, 1/3n\}$. Assume further that U misses E_c and E_d. Let V be a neighborhood of q such that $V \subset U$ and $V \cap \Sigma$ lies in an n-cell in $U \cap (\Sigma-(\Sigma_c \cup \Sigma_d))$. We must show that any map f of the boundary of a 2-cell D into $V \cap Z$ has an extension g sending D into a $(1/n)$-subset of Z.

Using the notation developed in [12] we trace the argument given on pp. 371–373 there with certain modifications. First, replace part A by the following observation: the hypotheses that $\Sigma_t=\text{Bd } Z_t$ and Z_t is 0-ULC imply that there exists an arc in $U \cap Z_t$ connecting each pair of points of $h(A_j^t) \cap f(\partial D)$. Let K_j^t denote the union of the (finitely many) arcs obtained in this way. Second, in part B observe that the special levels (the t's) can be chosen from Y, since Y is dense in $[c, d]$. The last two paragraphs of B may be ignored, noting instead that any map of a simple closed curve into

$$(K^{t_i} \cup K^{t_{i+1}} \cup h(\partial D)) \cap \bigcup \{E_t | t_i \leqq t \leqq t_{i+1}\}$$

is homotopic, in the intersection of Z and the $(1/3n)$-neighborhood of U, to a constant map. Making use of this fact at the end of part D, we can construct the required extension g sending D into Z.

COROLLARY 6.2. *Suppose that Σ is a closed n-manifold in E^{n+1}, U a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that for each $t \in (a, b)$, Σ_t is a closed $(n-1)$-manifold that is collared from U_t. Then (a, b) contains a dense G_δ-subset G such that U is* 1-*LC at each point of* $\Sigma(G)$.

THEOREM 6.3. *Suppose that Σ is a closed n-manifold in E^{n+1}, Z a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that* (i) $\text{Cl}(Z)-\Sigma_t$ *is* 1-*ULC and* (ii) Z_t *is* 1-*ULC for each t in (a, b). Then (a, b) contains a dense G_δ-subset G such that Z is* 1-*LC at each point of* $\Sigma(G)$.

Proof. We begin by repeating the first three paragraphs of the proof of Theorem 6.1. Then, with minor modifications similar to those given in the preceding proposition, Cannon's argument [7, Theorem 1] can be applied to complete the proof.

It would be interesting to know whether the hypotheses of either Theorem 6.1 or 6.3 actually imply that Z is 1-LC at each point of $\Sigma(a, b)$. One should note that some restrictions on the set Z are necessary, for it is quite simple to describe a connected open subset Z of E^{n+1} $(n \geqq 2)$ such that each Z_t is 1-ULC but Z fails to be 1-LC at certain points of Bd Z. In particular, a bounded open subset Z of E^{n+1} $(n \geqq 2)$ need not be 1-ULC even if each Z_t is 1-ULC.

THEOREM 6.4. *Suppose Σ is a closed n-manifold in E^{n+1}, U a component of $E^{n+1}-\Sigma$, and (a, b) an interval such that* (1) *for each $t \in (a, b)$, U_t is* 1-*ULC and* (2) *for each compact* 0-*dimensional subset C of (a, b),* $\text{Cl}\, U-\Sigma(C)$ *is* 1-*ULC. Then U is* 1-*LC at each point of* $\Sigma(a, b)$.

Proof. Let f be a map of the disk Δ^2 into Cl U such that $f(\partial\Delta^2) \subset U$ and $f(\Delta^2) \cap \Sigma \subset \Sigma(a, b)$. If G denotes the dense G_δ-subset of (a, b) promised by Theorem 6.3, then from hypothesis (2) above and Theorem 3.2 we find that f can be adjusted slightly, not changing the map on $\partial\Delta^2$, such that $f(\Delta^2) \cap \Sigma \subset \Sigma(G)$ and $f^{-1}(\Sigma \cap f(\Delta^2))$ is 0-dimensional. As a result, U is 1-LC at each point of $\Sigma \cap f(\Delta^2)$, and it is then a simple matter to alter f further so that $f(\Delta^2)$ misses Σ entirely. Hence, U is 1-LC at each point of $\Sigma(a, b)$.

COROLLARY 6.5. *Suppose Σ is a closed n-manifold in E^{n+1} such that each component U of $E^{n+1}-\Sigma$ satisfies* (1) *for each $t \in E^1$, U_t is* 1-*ULC and* (2) *for each compact,* 0-*dimensional subset C of E^1,* $\text{Cl}\, U-\Sigma(C)$ *is* 1-*ULC. Then $E^{n+1}-\Sigma$ is* 1-*ULC.*

Corollary 6.5 can be interpreted as a generalization of [7, Corollary 3].

REMARK. Variations on Theorem 6.4 and Corollary 6.5 can be obtained by exchanging condition (2) in each for the following condition:

(2*) for each compact 0-dimensional subset C of (a, b), or E^1, as the context requires, $\Sigma-\Sigma(C)$ is 1-ULC.

The proof given for Theorem 2 of [7] establishes that to each $\delta>0$ there corresponds an $\alpha>0$ such that each α-loop in U is contractible in an ε-subset of $E^{n+1}-\Sigma(C)$. Using this we can prove, as in Theorem 5.3, that $\mathrm{Cl}\, U-\Sigma(C)$ is 1-ULC.

Finally, Corollary 6.5 and [16, Theorem 9] can be combined, as in §5, to produce a local flatness criterion.

THEOREM 6.6. *Suppose Σ is a closed PL n-manifold in E^{n+1} $(n\geqq 4)$ satisfying the hypotheses of Corollary* 6.5. *Then Σ is locally flat if and only if Σ can be homeomorphically approximated by locally flat manifolds.*

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